Tutorial 10

An Alternative Definition

We present an alternative definition for the Poisson process.

Definition. A Poisson process $\{X(t)\}_{t\geq 0}$ with rate $\lambda > 0$ is a time homogeneous process satisfying Markov property with state space $\mathcal{S} = \mathbb{N}$, initial state $X(0) = 0$, and transition function

$$P_{xy}(t) = \begin{cases} \dfrac{(\lambda t)^{y-x} e^{-\lambda t}}{(y-x)!}, & 0 \leq x \leq y,\ t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

where the Markov property and the time homogeneity are defined as

Markov property. For any $0 \leq s_1 \leq \cdots s_n \leq s \leq t$ and any $x_1, \cdots, x_n, x, y \in \mathcal{S}$,

$$P(X(t) = y \mid X(s_1) = x_1, \cdots, X(s_n) = x_n, X(s) = x) = P(X(t) = y \mid X(s) = x).$$

Time homogeneity. For any $0 \leq s \leq t$ and any $x, y \in \mathcal{S}$,

$$P(X(t) = y \mid X(s) = x) = P(X(t-s) = y \mid X(0) = x) = P_x(X(t-s) = y).$$

If the process $X(t)$ satisfies above two properties, one can define the *transition function* as follows

$$P_{xy}(t) = P_x(X(t) = y), \quad t \geq 0,\ x, y \in \mathcal{S}.$$

We claim that the above definition is equivalent to the one in the textbook. To verify this claim, it suffices to prove that $X(t)$ satisfies the following three properties:

(i) (*Initial state*) $X(0) = 0$;

(ii) (*Stationary Poisson increments*) $X(t) - X(s)$ has a Poisson distribution with parameter $\lambda(t-s)$ for $0 \leq s \leq t$.

(iii) (*Indenpendent increments*) $X(t_2) - X(t_1),\ X(t_3) - X(t_2),\ \cdots,\ X(t_n) - X(t_{n-1})$ are independent for $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$.

Proof. (i) is directly given in definition.

(ii) By formula (1), one can see

$$P_{xy}(t) = P_{0,y-x}(t), \quad t \geq 0. \tag{2}$$

Hence for $0 \leq s \leq t$ and $y \geq 0$,

$$\begin{aligned} P(X(t) - X(s) = y) &= \sum_{x=0}^{\infty} P(X(s) = x, X(t) = x + y) \\ &= \sum_{x=0}^{\infty} P(X(s) = x) P(X(t) = x + y \mid X(s) = x) \\ &= \sum_{x=0}^{\infty} P(X(s) = x) P_{x,x+y}(t-s) \quad \text{(by time homogeneity)} \\ &= \sum_{x=0}^{\infty} P(X(s) = x) P_{0,y}(t-s) \qquad \text{(by (2))} \\ &= P_{0,y}(t-s) \qquad\qquad\qquad (3) \\ &= \frac{(\lambda(t-s))^y e^{-\lambda(t-s)}}{y!}. \end{aligned}$$

(iii) For $0 \le t_1 \le t_2 \le \cdots \le t_n$ and $z_1, z_2, \cdots, z_{n-1} \in \mathbb{N}$,

$$\begin{aligned}
& P(X(t_2)-X(t_1)=z_1, X(t_3)-X(t_2)=z_2, \cdots, X(t_n)-X(t_{n-1})=z_{n-1}) \\
= & \sum_{x=0}^{\infty} P(X(t_1)=x) P(X(t_2)-X(t_1)=z_1, X(t_3)-X(t_2)=z_2, \cdots, \\
& X(t_n)-X(t_{n-1})=z_{n-1} \mid X(t_1)=x) \\
= & \sum_{x=0}^{\infty} P(X(t_1)=x) P_{x,x+z_1}(t_2-t_1) P_{x+z_1,x+z_1+z_2}(t_3-t_2) \cdots \\
& P_{x+z_1+\cdots+z_{n-2},x+z_1+\cdots+z_{n-1}}(t_n-t_{n-1}) \quad \text{(by Markov property and time homogeneity)} \\
= & \sum_{x=0}^{\infty} P(X(t_1)=x) P_{0,z_1}(t_2-t_1) P_{0,z_2}(t_3-t_2) \cdots P_{0,z_{n-1}}(t_n-t_{n-1}) \quad \text{(by (2))} \\
= & P_{0,z_1}(t_2-t_1) P_{0,z_2}(t_3-t_2) \cdots P_{0,z_{n-1}}(t_n-t_{n-1}) \\
= & P(X(t_2)-X(t_1)=z_1) P(X(t_3)-X(t_2)=z_2) \cdots P(X(t_n)-X(t_{n-1})=z_{n-1}).
\end{aligned}$$

The last step follows from formula (3).

Remark. This alternative definition will appear to be more convenient for use in practice.

Sum of independent Poisson processes

Two continuous-time process $\{X_1(s)\}_{s\ge 0}$ and $\{X_2(t)\}_{t\ge 0}$ are said to be *independent* if for any time instants $s, t \ge 0$, random variables $X_1(s)$ and $X_2(t)$ are independent.

Theorem 1. Suppose that $X_1(t)$ and $X_2(t)$ are independent Poisson processes with rates λ and μ. Then $X_1(t) + X_2(t)$ is a Poisson process with rate $\lambda + \mu$.

Proof. The conclusion holds if and only if the process $X_1(t) + X_2(t)$ starts with initial state 0 and has the stationary and independent increments with certain Poisson distributions.

The initial state is $X_1(0) + X_2(0) = 0$ clearly. Since both $X_1(t)$ and $X_2(t)$ have independent increments, so does $X_1(t) + X_2(t)$. For time instants $s, t \ge 0$, let $Y = X_1(t+s) - X_1(s)$ and $Z = X_2(t+s) - X_2(s)$ be the increments. Then

$$\begin{aligned}
P(Y+Z=n) &= \sum_{m=0}^{n} P(Y=m) P(Z=n-m) \\
&= \sum_{m=0}^{n} e^{-\lambda t} \frac{(\lambda t)^m}{m!} \cdot e^{-\mu t} \frac{(\mu t)^{n-m}}{(n-m)!} \\
&= \frac{e^{-(\lambda+\mu)t}}{n!} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} (\lambda t)^m (\mu t)^{n-m} \\
&= e^{-(\lambda+\mu)t} \frac{\left((\lambda+\mu)t\right)^n}{n!}.
\end{aligned}$$

Hence the increment $X_1(t+s) + X_2(t+s) - X_1(s) - X_2(s) \sim \mathrm{Poi}((\lambda+\mu)t)$.

Remark. One can generalize above theorem to the sum of n independent Poisson processes by induction.

Decomposition of Poisson process

In some concrete situation of Poisson processes, the arrivals can be seen in different types. For example, in a soccer game between Team A and Team B, we may assume that the goals just follow a Poisson process.

Most soccer fans really focus on which team makes each goal so may predict the results in halftime or in final. Some fans may only be interested in the goals made by Team A. Whatever, we could have a way to decompose the process into two parts as subprocesses.

Theorem 2. Suppose that $X(t)$ is a Poisson process with rate λ. There are two types of arrivals. For each arrival, it has probability p $(0 < p < 1)$ to be type-1 and probability $q = 1 - p$ to be type-2. Let $X_i(t)$ be the number of arrivals in type-i by time t, $i = 1, 2$. Then $X_1(t)$ and $X_2(t)$ are independent Poisson processes with rates λp and λq.

Proof. The idea is similar to the proof of Theorem 1.

The initial states are $X_1(0) = 0 = X_2(0)$ clearly. Since $X(t)$ has independent increments, so do $X_1(t)$ and $X_2(t)$. For time instant $s, t \geq 0$, let $Y = X_1(t+s) - X_1(s)$ and $Z = X_2(t+s) - X_2(s)$ be the increments. Then we need to prove that Y and Z are independent Poisson distributed with parameters λpt and λqt. For $j, k \in \mathbb{N}$,

$$\begin{aligned} P(Y = j, Z = k) &= P(Y = j \mid Y + Z = j + k) P(Y + Z = j + k) \\ &\quad \big(\text{under the condition } Y + Z = j + k, Y \sim B(j + k, p)\big) \\ &= \frac{(j+k)!}{j!k!} p^j q^k \cdot e^{-\lambda t} \frac{(\lambda t)^{j+k}}{(j+k)!} \\ &= e^{-\lambda pt} \frac{(\lambda pt)^j}{j!} e^{-\lambda qt} \frac{(\lambda qt)^k}{k!}. \end{aligned} \tag{4}$$

Hence $P(Y = j) = \sum_{k=0}^{\infty} P(Y = j, Z = k) = e^{-\lambda pt} \dfrac{(\lambda pt)^j}{j!}$ which implies that $Y \sim \text{Poi}(\lambda pt)$. Similarly, $Z \sim \text{Poi}(\lambda qt)$. Moreover, formula (4) implies that $P(Y = j, Z = k) = P(Y = j)P(Z = k)$. Thus Y and Z are independent.

Remark. One can also generalize above theorem to the situation of n different types of arrivals by induction.